Math 126 Summer 2026 HW 3

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Problem 1: (Borthwick 7.6) On HW 3, we investigated a quantity

$$\eta(t) = \int_\Omega u(t,x)^2 dx$$

for u a smooth solution to the heat equation on $(0,\infty)\times\Omega$ with bounded C^1 domain Ω. Suppose u is such a solution and $u|_{t=T} = 0$, $u|_{\partial\Omega} = 0$.

Part A.) Use Cauchy-Schwarz to deduce that

$$\eta'(t)^2 \leq 4\eta(t)\int_\Omega |\frac{\partial u}{\partial t}|^2 dx$$

Part B.) Show

$$\eta''(t) = 4\int_\Omega |\frac{\partial u}{\partial t}|^2 dx$$

so that from A.), $\eta'(t)^2 \leq \eta(t)\eta''(t)$.

Part C.) Suppose $\eta(0) > 0$ and η is defined in some neighborhood of 0. Using part B.), show

$$(\log(\eta(t))'' \geq 0$$

This implies that $\log(\eta(t))$ is *convex* and, in particular, bounded below by its tangent lines such that

$$\log(\eta(t)) \geq \log(\eta(0)) + \frac{\eta'(0)}{\eta(0)}t$$

so that $\eta(t) \geq \eta(0)e^{-ct}$. Thus, if $\eta(0) > 0$, η is strictly positive for all time.

Part D.) Conclude from Part C.) that if $\eta(T) = 0$, $\eta(t) = 0$ for all t and $u = 0$ (it may help to reference HW 3).

Problem 2: In the separation of variables section, we used the Legendre polynomials to help construct solutions. These are an orthonormal basis for L^2 via polynomials, as opposed to exponentials.

Part A.) Use Gram-Schmidt on $\{1, x, x^2\}$ with the $L^2((-1,1))$-inner product to obtain the first 3 Legendre polynomials. Verify that these are normalized (norm 1) versions of the first three solutions of

$$P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$$

Part B.) These polynomials are solutions of the ODE

$$\frac{d}{dx}\left[(x^2-1)\frac{d}{dx}f\right] - k(k+1)f = 0$$

or eigenvectors $Lf = k(k+1)f$ for $L = \frac{d}{dx}\left[(x^2-1)\frac{d}{dx}\right]$. For $u, v \in C^2([-1,1])$, check that

$$\langle u, Lv\rangle = \langle Lu, v\rangle$$

so that the P_k with distinct k are orthogonal.

Part C.) Let $f(x) \in L^2((-1,1))$ be a function so

$$\int_{-1}^{1} f(x) x^l dx = 0$$

for $l = 0, 1, 2, ...$ Show that

$$\int_{-1}^{1} f(x) S_m(x) dx = 0$$

for $S_m(x) = \sum_{l=0}^{m} \frac{(-ikx)^l}{l!}$. Since $\lim_{m\to\infty} S_m(x) = e^{-ikx}$, show f must be 0.

Part D.) Use Part C.) to show the Legendre polynomial P_k are a basis for L^2.